Серебряков Артём
ФН11-53Б
Вариант 11
Постановка задачи
Найти свертку функций f(x) и g(x)
Найдем уравнения прямых на заданных отрезках для функции f(x)
A(-2,-2) B(0,-2) C(3,2) D(4,0)
На AB: f(x)≡-2
На BC:

$$solve\left(\frac{(y+2)}{2-(-2)} = \frac{(x-0)}{3-0}, y\right)$$

$$\frac{4}{3}x - 2 \tag{1}$$

На CD:

$$solve\left(\frac{(y-2)}{0-2} = \frac{(x-3)}{4-3}, y\right)$$

$$-2x + 8 \tag{2}$$

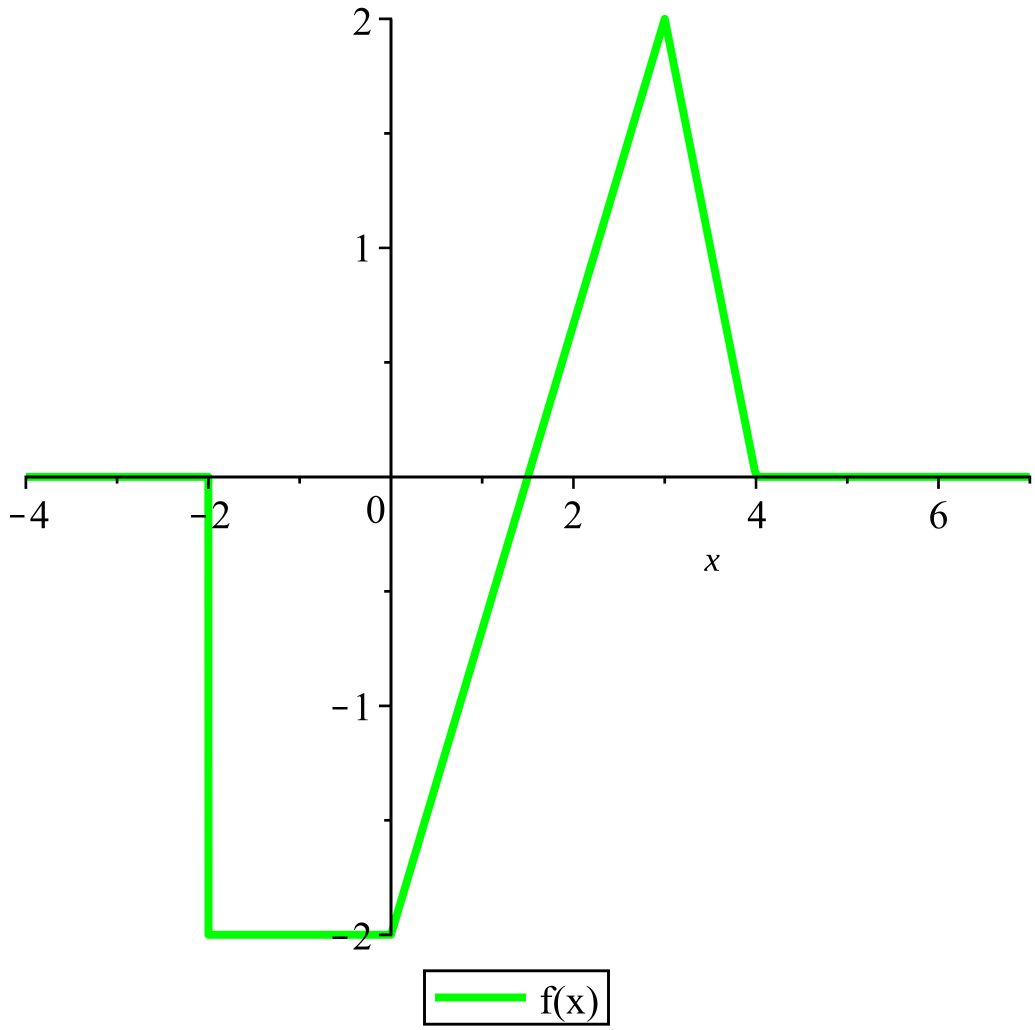
Получаем:

$$f(x) := piecewise\left(x < -2, 0, -2 \le x \le 0, -2, 0 \le x \le 3, \frac{4}{3}x - 2, 3 \le x \le 4, -2x + 8, x > 4, 0\right) :$$

$f(x)$

$$\begin{cases} 0 & x < -2 \\ -2 & -2 \le x \textbf{ and } x \le 0 \\ \frac{4}{3}x - 2 & 0 \le x \textbf{ and } x \le 3 \\ -2x + 8 & 3 \le x \textbf{ and } x \le 4 \\ 0 & 4 < x \end{cases} \tag{3}$$

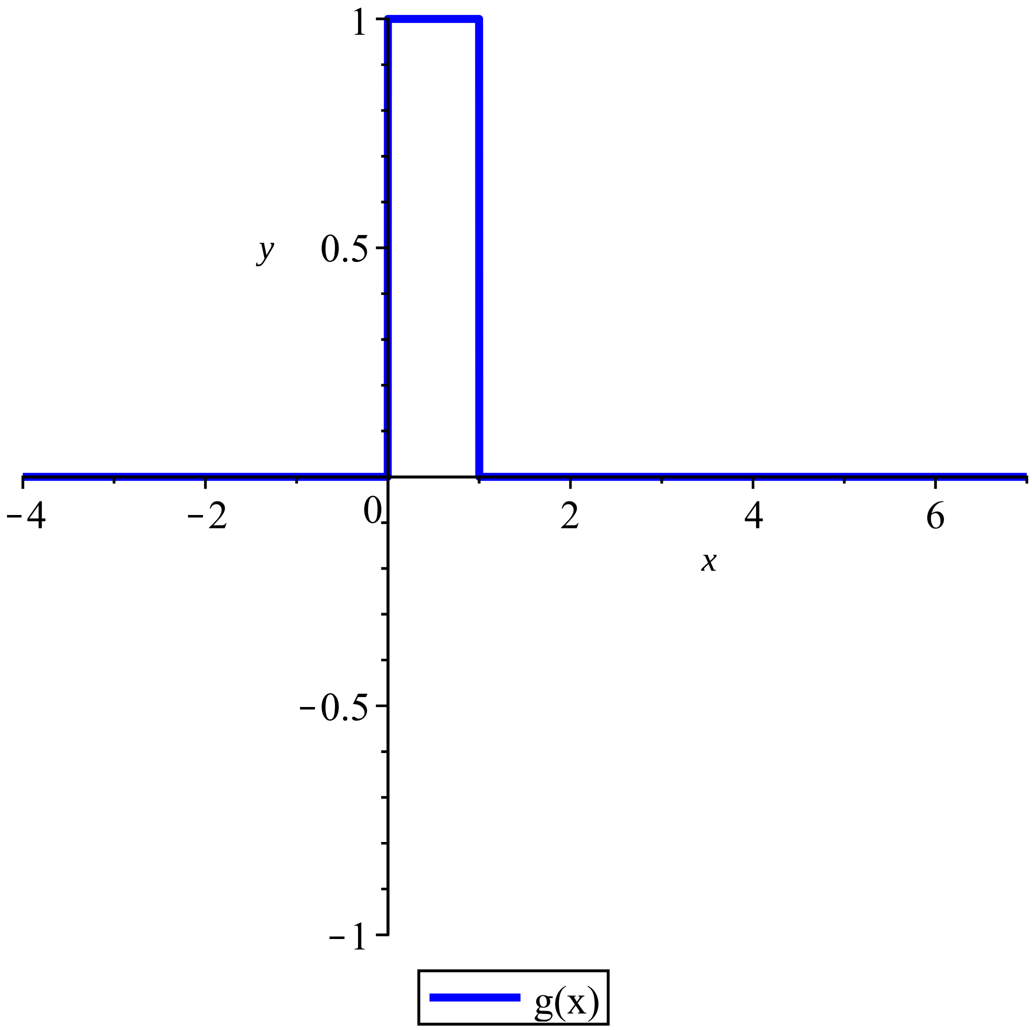
$plot(f(x), x = -4..7, legend = \text{"f(x)"}, color = green, thickness = 3)$



$g(x) := \mathit{piecewise}(x < 0, 0, 0 \le x < 1, 1, x \ge 1, 0)$:
$g(x)$

$$\begin{cases} 0 & x < 0 \\ 1 & 0 \le x \ \textbf{and} \ x < 1 \\ 0 & 1 \le x \end{cases} \tag{4}$$

$\mathit{plot}(g(x), x = -4\,..7, y = -1\,..1, \mathit{legend} = \text{"g(x)"}, \mathit{color} = \mathit{blue}, \mathit{thickness} = 3)$



$f(\text{tau})$

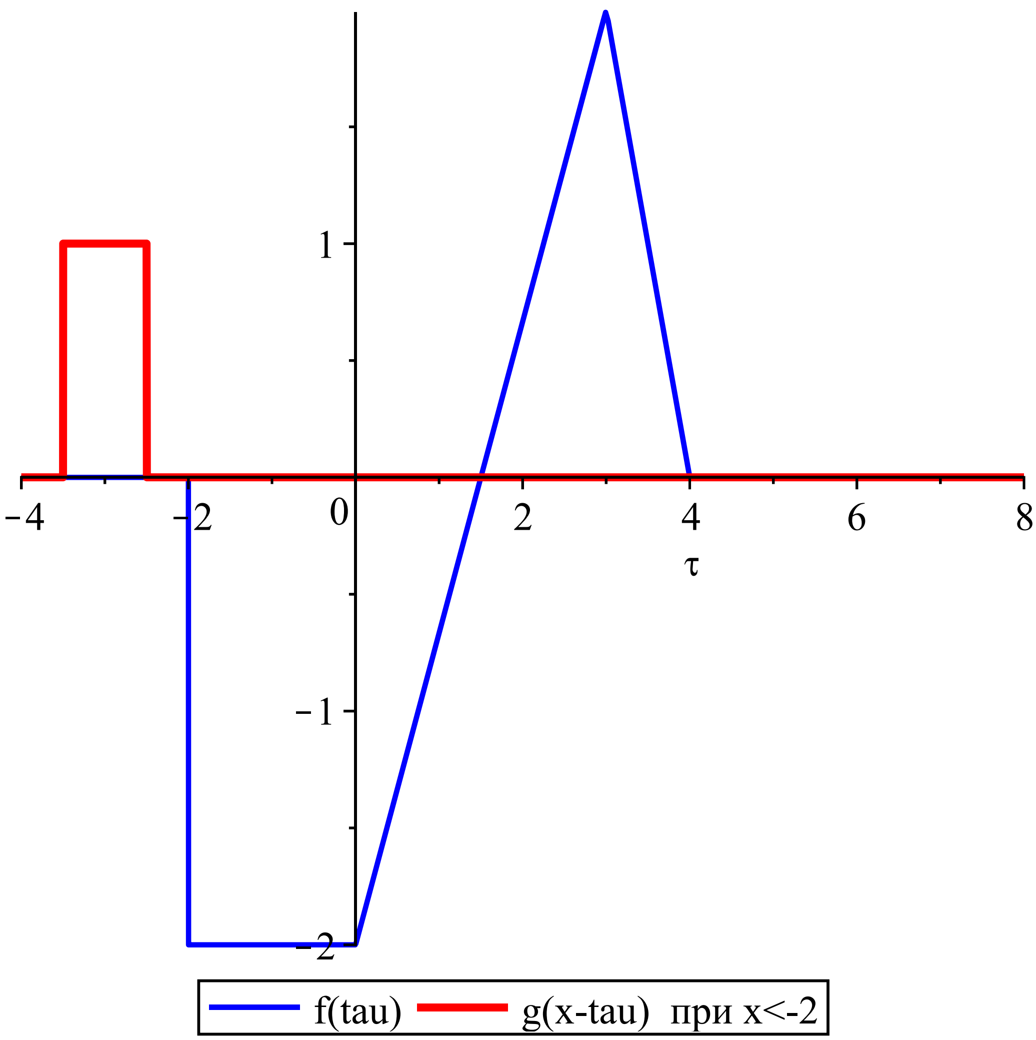
$$\begin{cases} 0 & \tau < -2 \\ -2 & -2 \le \tau \textbf{ and } \tau \le 0 \\ \frac{4}{3}\tau - 2 & 0 \le \tau \textbf{ and } \tau \le 3 \\ -2\tau + 8 & 3 \le \tau \textbf{ and } \tau \le 4 \\ 0 & 4 < \tau \end{cases} \quad \textbf{(5)}$$

$g(x - \text{tau}) := \mathit{piecewise}(\text{tau} \le x - 1, 0, \text{tau} > x - 1 \textbf{ and } \text{tau} \le x, 1, \text{tau} > x, 0)$

$$\begin{cases} 0 & \tau \le -1 + x \\ 1 & -1 + x < \tau \textbf{ and } \tau \le x \\ 0 & x < \tau \end{cases} \quad \textbf{(6)}$$

1) x<-2

plot([*f*(tau), *eval*(*g*(*x* − tau), *x* =−2.5)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при x<-2"]);



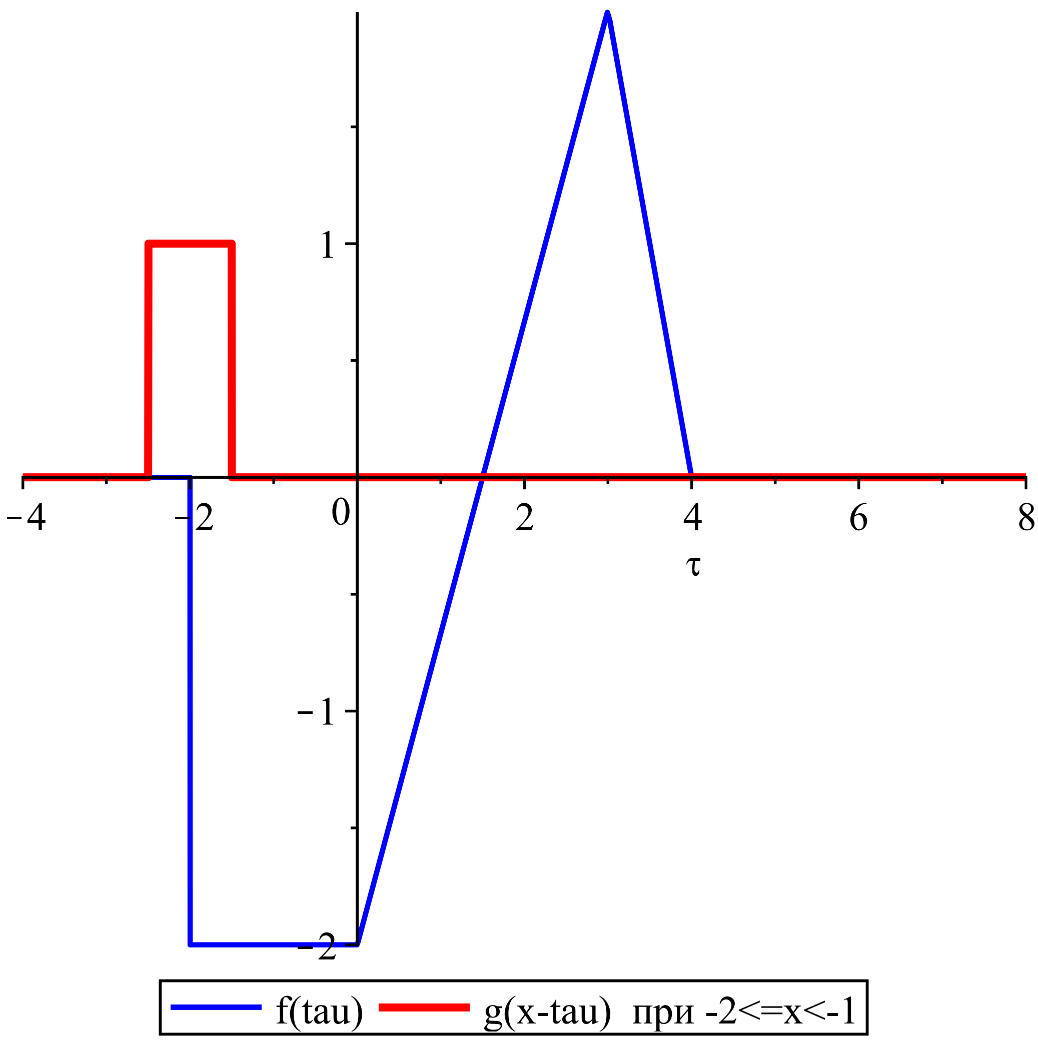
$sv1(x) := \int_{x-1}^{x} 0 \, \mathrm{dtau}$:

$sv1(x)$

$$0 \tag{7}$$

2)-2<=x<-1

plot([*f* (tau), *eval*(*g* (*x* − tau), *x* =−1.5)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при -2 <=x <-1"])



$sv2(x) := \int_{x-1}^{-2} 0 \, \mathrm{dtau} + \int_{-2}^{x} 1 \cdot (-2) \, \mathrm{dtau} :$

$sv2(x)$

$$-2\,x - 4 \qquad \textbf{(8)}$$

$sv2(-2)$

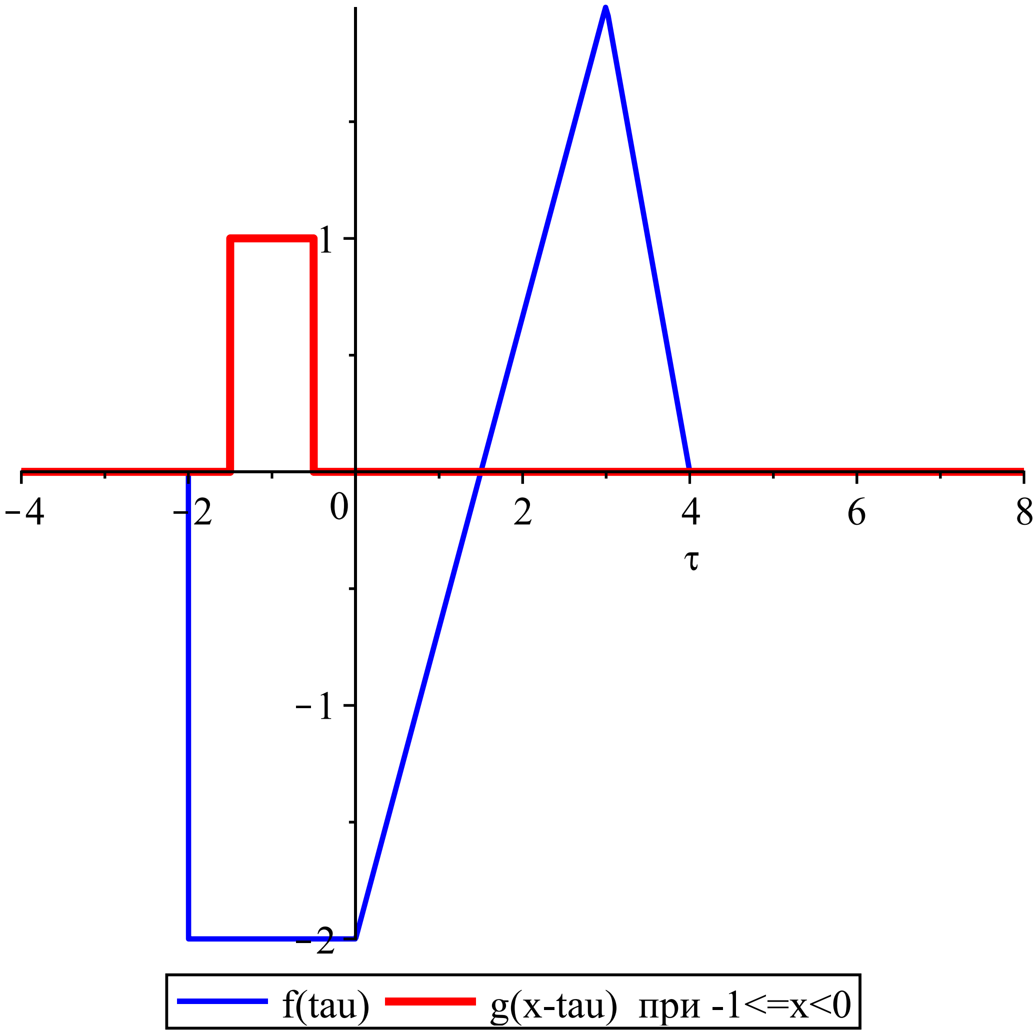
$$0 \qquad \textbf{(9)}$$

$sv2(-1)$

$$-2 \qquad \textbf{(10)}$$

3)-1<=x<0

plot([*f* (tau), *eval*(*g*(*x* − tau), *x* =−0.5)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при -1<=x<0"])



$sv3(x) := \int_{x-1}^{x} 1 \cdot (-2) \, \mathrm{dtau}:$

$sv3(x)$

$$-2 \tag{11}$$

$sv3(-1)$

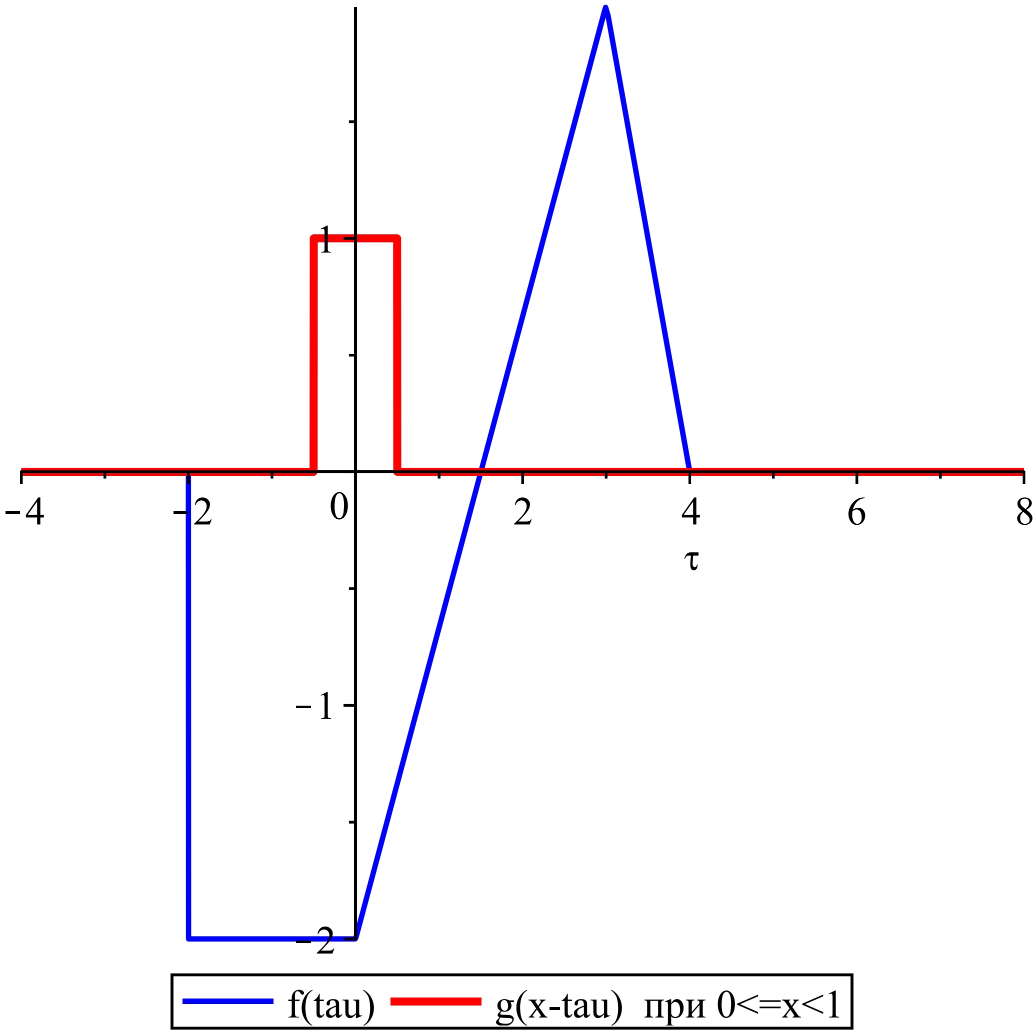
$$-2 \tag{12}$$

$sv3(0)$

$$-2 \tag{13}$$

4)0<=x<1

plot([*f* (tau), *eval*(*g*(*x* − tau), *x* = 0.5)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при 0 <= x < 1"])



$$sv4(x) := \int_{x-1}^{0} 1 \cdot (-2)\, \mathrm{dtau} + \int_{0}^{x} 1 \cdot \left(\frac{4}{3} \cdot \mathrm{tau} - 2\right) \mathrm{dtau} :$$

$sv4(x)$

$$-2 + \frac{2}{3} x^2 \quad (14)$$

$sv4(0)$

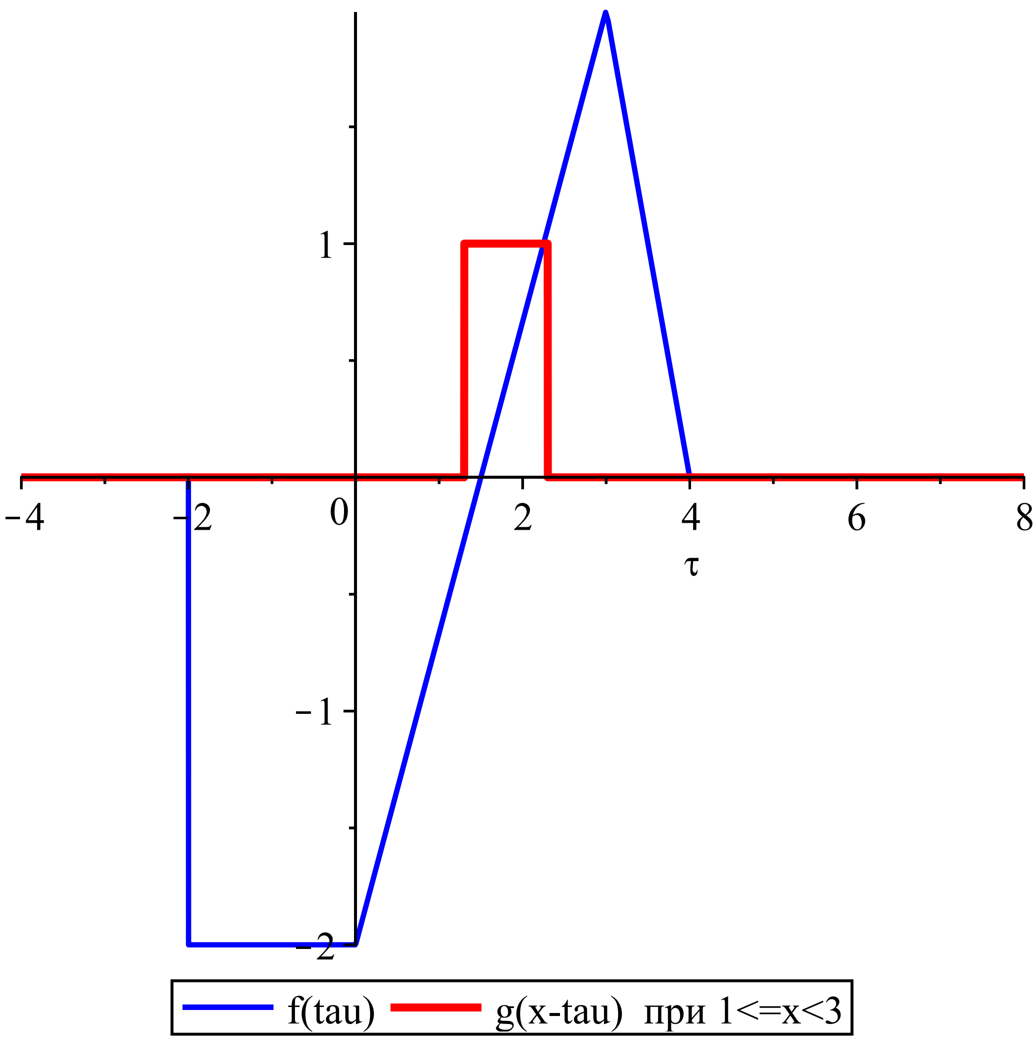
$$-2 \quad (15)$$

$sv4(1)$

$$-\frac{4}{3} \quad (16)$$

5)1<=x<3

plot([*f* (tau), *eval*(*g* (*x* − tau), *x* = 2.3)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при 1<=x<3"])



$sv5(x) := \int_{x-1}^{x} 1 \cdot \left(\frac{4}{3} \cdot \mathrm{tau} - 2\right) \mathrm{dtau}:$

$sv5(x)$

$$\frac{2}{3}x^2 - \frac{2}{3}(-1+x)^2 - 2 \tag{17}$$

$sv5(1)$

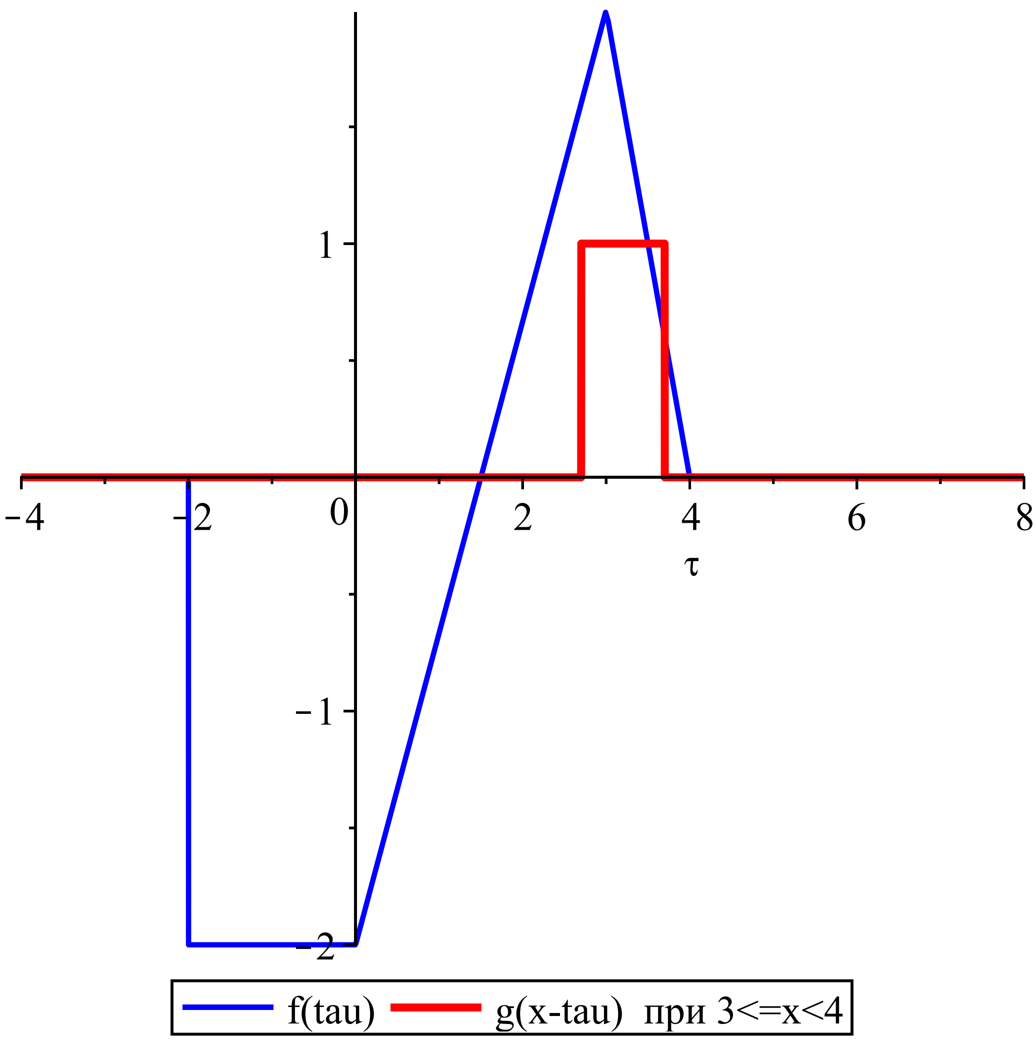
$$-\frac{4}{3} \tag{18}$$

$sv5(3)$

$$\frac{4}{3} \tag{19}$$

6)3<=x<4

plot([*f* (tau), *eval*(*g* (*x* − tau), *x* = 3.7)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при 3 <=x <4"])



$$sv6(x) := \int_{x-1}^{3} 1\cdot\left(\frac{4}{3}\cdot \mathrm{tau} - 2\right) \mathrm{dtau} + \int_{3}^{x} 1\cdot(-2\cdot \mathrm{tau} + 8)\,\mathrm{dtau}:$$

$sv6(3)$

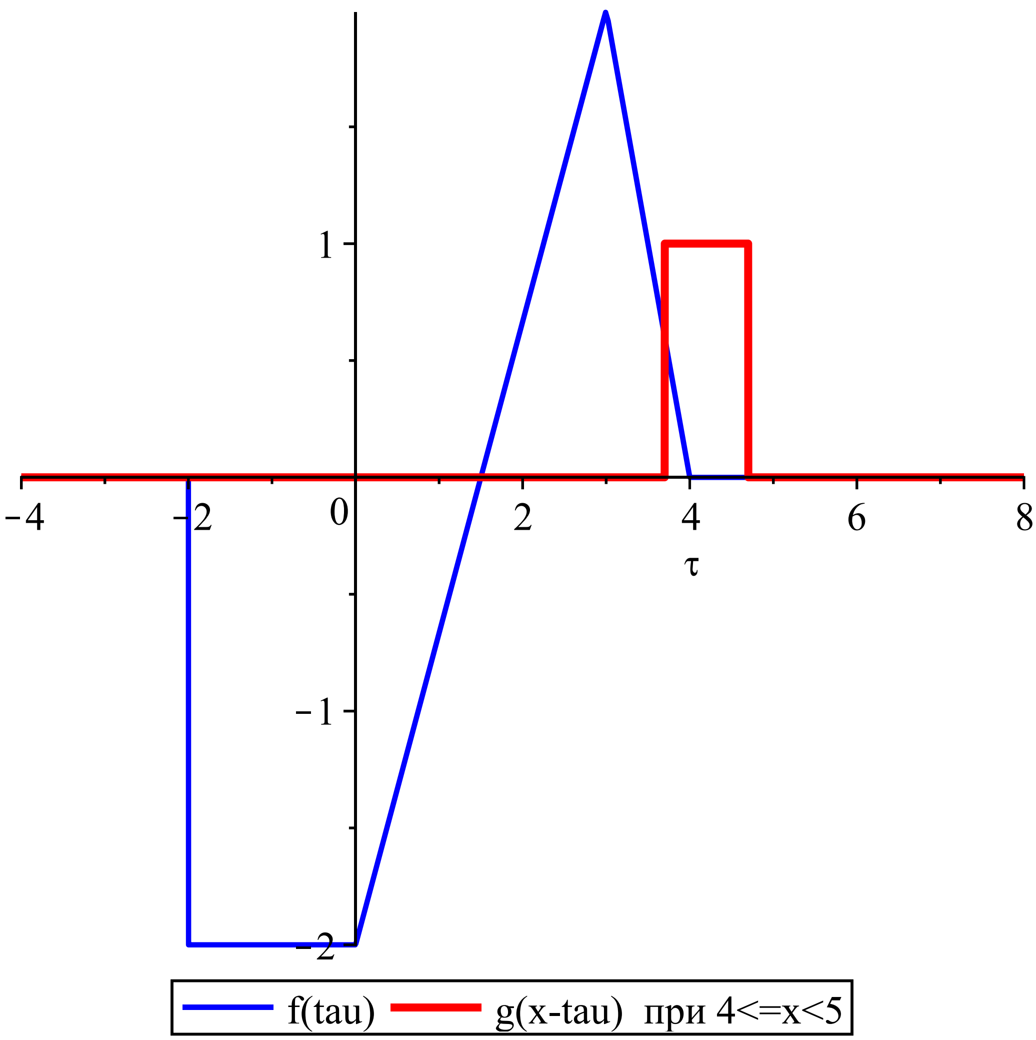
$$\frac{4}{3} \qquad \textbf{(20)}$$

$sv6(4)$

$$1 \qquad \textbf{(21)}$$

7)4<=x<5

plot([*f* (tau), *eval*(*g* (*x* − tau), *x* = 4.7)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при 4<=x<5"])



$sv7(x) := \int_{x-1}^{4} 1 \cdot (-2 \cdot \mathrm{tau} + 8)\, d\mathrm{tau}:$

$sv7(4)$

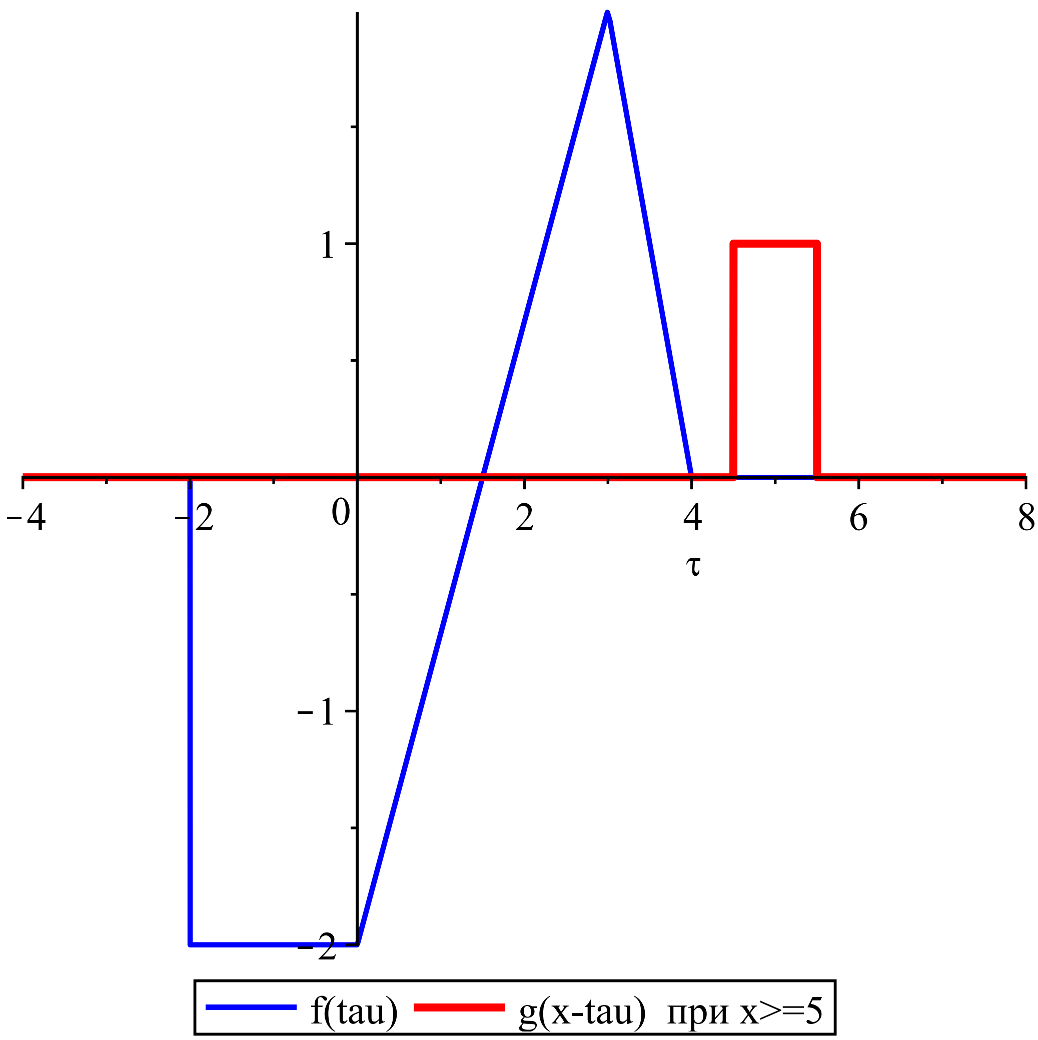
$$1 \tag{22}$$

$sv7(5)$

$$0 \tag{23}$$

8)x>=5

plot([*f* (tau), *eval*(*g* (*x* − tau), *x* = 5.5)], tau =−4 ..8, *thickness* = [2, 3], *color* = [*blue*, *red*], *legend* = ["f(tau)" , "g(x-tau) при x>=5"])



$sv8(x) := 0 :$
$sv8(x)$

$$0 \tag{24}$$

$sv8(5)$

$$0 \tag{25}$$

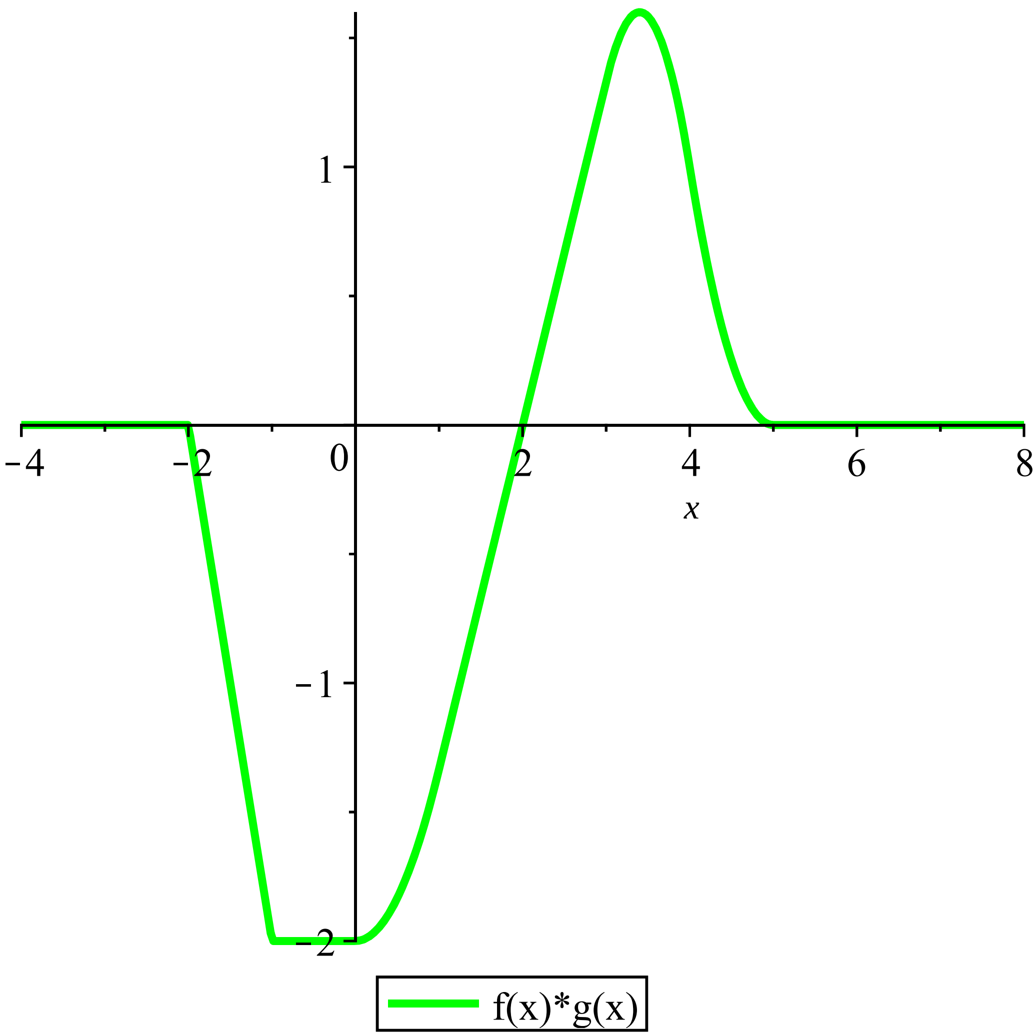
График полученной функции

$sv(x) := piecewise(\ x < -2, sv1(x), x \geq -2 \textbf{ and } x < -1, sv2(x), x \geq -1 \textbf{ and } x < 0, sv3(x), x \geq 0 \textbf{ and } x < 1, sv4(x), x \geq 1 \textbf{ and } x < 3, sv5(x), x \geq 3 \textbf{ and } x < 4, sv6(x), x \geq 4 \textbf{ and } x < 5, sv7(x), x \geq 5, sv8(x)\ ) :$

$sv(x)$

$$\begin{cases} 0 & x < -2 \\ -2\,x-4 & -2 \le x \textbf{ and } x < -1 \\ -2 & -1 \le x \textbf{ and } x < 0 \\ -2+\frac{2}{3}\,x^2 & 0 \le x \textbf{ and } x < 1 \\ \frac{2}{3}\,x^2-\frac{2}{3}\,(-1+x)^2-2 & 1 \le x \textbf{ and } x < 3 \\ -17-\frac{2}{3}\,(-1+x)^2+10\,x-x^2 & 3 \le x \textbf{ and } x < 4 \\ 24+(-1+x)^2-8\,x & 4 \le x \textbf{ and } x < 5 \\ 0 & 5 \le x \end{cases} \tag{26}$$

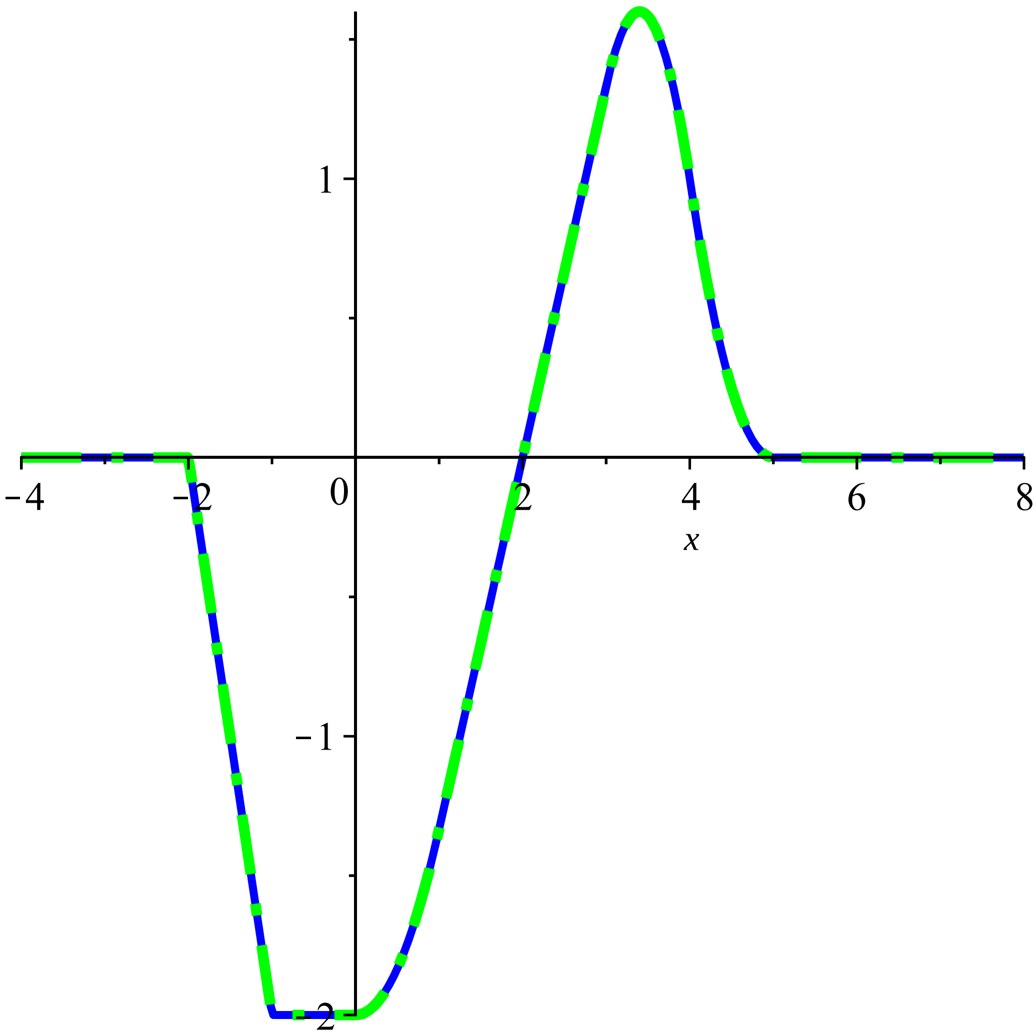
$plot(sv(x), x=-4..8, color=green, thickness=3, legend="\text{f(x)*g(x)}")$



Проверка

$$FG(x) := \int_{-\infty}^{+\infty} f(\mathrm{tau})\cdot g(x-\mathrm{tau})\,\mathrm{dtau}:$$

$plot([sv(x), FG(x)], x = -4..8, color = [blue, green], linestyle = [1, 4], thickness = [3, 4])$



Найти образ Фурье функции f(x), если f(x)≡0 при x∉[x1,x4], а при x∈[x1,x4] график этой функции состоит из звеньев ломанной, проходящей через точки
A(-1,1)
B(0,1)
C(1,-3)
D(2,1)
На AB: f(x)≡1
На BC:

$$solve\left(\frac{(y-1)}{-3-1} = \frac{(x-0)}{1-0}, y\right)$$

$$-4\,x+1 \qquad \textbf{(27)}$$

На CD:

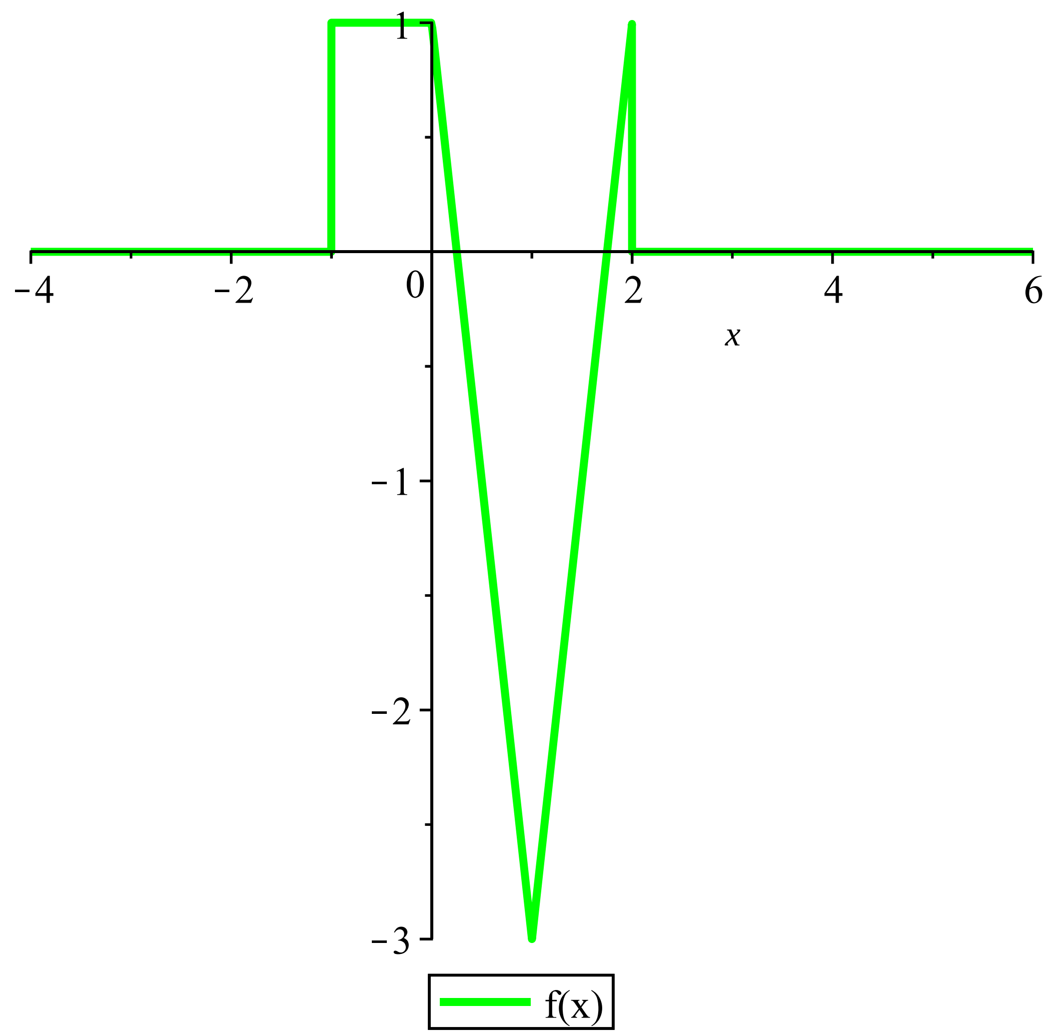
$$solve\left(\frac{(y-(-3))}{1-(-3)} = \frac{(x-1)}{2-1}, y\right)$$

$$-7+4\,x \qquad \textbf{(28)}$$

Получаем:
restart :
$f(x) := piecewise(x \le -1, 0, -1 < x \le 0, 1, 0 < x \le 1, -4\,x+1, 1 < x \le 2, 4\,x-7, x > 2, 0)$:
$f(x)$

$$\begin{cases} 0 & x \leq -1 \\ 1 & -1 < x \textbf{ and } x \leq 0 \\ -4\,x+1 & 0 < x \textbf{ and } x \leq 1 \\ 4\,x-7 & 1 < x \textbf{ and } x \leq 2 \\ 0 & 2 < x \end{cases} \tag{29}$$

$plot(\,f(x), x=-4..6, legend = \text{"f(x)"}, color = green, thickness = 3\,)$



Представим f(x) в виде суммы треугольного и прямоугольного импульса

$rect(x) := piecewise\left(|x| < \frac{1}{2}, 1, |x| > \frac{1}{2}, 0\right)$

$$x \rightarrow piecewise\left(|x| < \frac{1}{2}, 1, \frac{1}{2} < |x|, 0\right) \tag{30}$$

$tri(x) := piecewise(-1 \leq x < 0, 1+x, 0 \leq x \leq 1, 1-x, |x| > 1, 0)$

$$x \rightarrow piecewise(-1 \leq x \textbf{ and } x < 0, 1+x, 0 \leq x \textbf{ and } x \leq 1, 1-x, 1 < |x|, 0) \tag{31}$$

f1(x)-прямоугольный импульс в интервале [-1,0]

$a := -1 :$

$b := 0 :$

$$m := \frac{a+b}{2}$$

$$-\frac{1}{2} \tag{32}$$

$$d := b - a$$

$$1 \tag{33}$$

$$f1(x) := rect\left(\frac{x-m}{d}\right):$$

$$f1(x)$$

$$\begin{cases} 1 & \left|x+\frac{1}{2}\right| < \frac{1}{2} \\ 0 & \frac{1}{2} < \left|x+\frac{1}{2}\right| \end{cases} \tag{34}$$

f2(x)-треугольный импульс в интервале [0,2]

$a1 := 0:$

$b1 := 2:$

$$m1 := \frac{a1+b1}{2}$$

$$1 \tag{35}$$

$$d1 := \frac{(b1-a1)}{2}$$

$$1 \tag{36}$$

$$c1 := -4$$

$$-4 \tag{37}$$

$$f2(x) := c1 \cdot tri\left(\frac{x-m1}{d1}\right):$$

f3(x)-прямоугольный импульс в интервале [0,2]

$$m2 := \frac{a1+b1}{2}$$

$$1 \tag{38}$$

$$d2 := b1 - a1$$

$$2 \tag{39}$$

$$f3(x) := rect\left(\frac{x-m2}{d2}\right)$$

$$x \rightarrow rect\left(\frac{x-m2}{d2}\right) \tag{40}$$

Получается, что f(x) - сумма двух прямоугольных и одного треугольного импульса

$fff(x) := f1(x) + f2(x) + f3(x):$

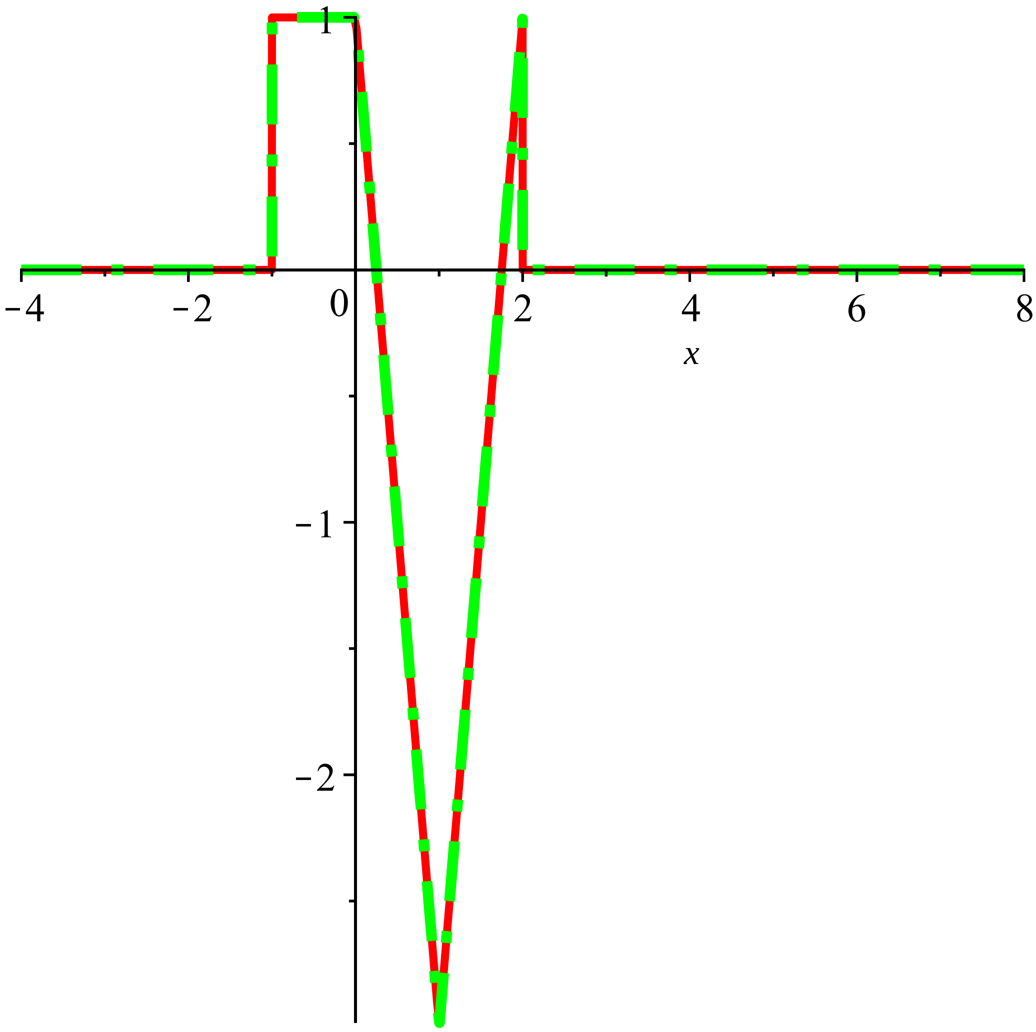
$fff(x)$

$$\left(\begin{cases} 1 & \left|x+\frac{1}{2}\right| < \frac{1}{2} \\ 0 & \frac{1}{2} < \left|x+\frac{1}{2}\right| \end{cases}\right) - 4\left(\begin{cases} x & 0 \le x \textbf{ and } x < 1 \\ 2-x & 0 \le -1+x \textbf{ and } x \le 2 \\ 0 & 1 < |-1+x| \end{cases}\right) + \Bigg(\tag{41}$$

$$\begin{cases} 1 & \left|-\frac{1}{2}+\frac{1}{2}x\right| < \frac{1}{2} \\ 0 & \frac{1}{2} < \left|-\frac{1}{2}+\frac{1}{2}x\right| \end{cases}$$

Проверка

$plot([f(x), fff(x)], x=-4..8, thickness=[3,4], color=[red, green], linestyle=[1,4])$



Образ Фурье f(x):

$\#F[f](v) = e^{i\cdot\pi v}\cdot sinc(\pi v) - 4\cdot e^{-i\cdot 2\cdot\pi v}\cdot sinc^2(\pi v) + 2\cdot e^{-i\cdot 2\cdot\pi v}\cdot sinc(2\cdot\pi v)$

$$F[f](v) := d\cdot\frac{e^{-I\pi v\cdot 2\cdot m}\sin(d\cdot\pi v)}{d\cdot\pi v} + c1\cdot d1\cdot\frac{e^{-2I\pi v\cdot m1}\sin^2(d1\cdot\pi v)}{d1^2\cdot\pi^2 v^2} + d2\cdot\frac{e^{-2I\pi v\cdot m2}\sin(\pi v\cdot d2)}{\pi v\cdot d2}:$$

$F[f](v)$

$$\frac{e^{I\pi v}\sin(\pi v)}{\pi v} - \frac{4\,e^{-2I\pi v}\sin(\pi v)^2}{\pi^2 v^2} + \frac{e^{-2I\pi v}\sin(2\pi v)}{\pi v} \qquad \textbf{(42)}$$